Cepstrum Analysis

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B.P. Bogert, M.J. R. Healy and J.W. Tukey, "The Quefrency Alanysis of Time Series for Echoes: Cepstrum, Psuedo-Autocovariance, Cross-cepstrum and Saphe Cracking", Proceedings of the Symposium on Time Series Analysis, M. Rosenblat, Ed., Wiley, NY, 1963, pp 209-243

Derived Terms	Original Terms
Cepstrum	Spectrum
Quefrency	Frequency
Rahmonics	Harmonics
Gamnitude	Magnitude
Saphe	Phase
Lifter	Filter
Short-pass Lifter	Low-pass Filter
Long-pass Lifter	High-pass Filter

original definition - real cepstrum

Inverse Fourier transfrom of the logarithm of the magnitude of the Fourier transform:

$$c_x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left| X\left(e^{j\omega}\right) \right| e^{j\omega n} d\omega$$

magnitude is real and nonnegative

not invertible as the phase is missing

more general definition complex cepstrum

$$\hat{x}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log[X(e^{j\omega})] e^{j\omega n} \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} [\log\left|X(e^{j\omega})\right| + j \arg(X(e^{j\omega}))] e^{j\omega n} \, d\omega$$

arg is the continuous (unwrapped) phase function.

The term complex cepstrum refers to the use of the complex logarithm, not to the sequence.

The complex cepstrum of a real sequence is also a real sequence.

complex logarithm

The complex cepstrum exists if the complex logarithm has a convergent power series representation of the form :

$$\hat{X}(z) = \log[X(z)] = \sum_{n=-\infty}^{\infty} \hat{x}[n]\, z^{-n} \quad , \qquad |z| = 1$$

So log[X(z)] must have the properties of the z-transform of a stable sequence.

using the inverse z-transform integral:

$$\hat{x}[n] = \frac{1}{2\pi j} \oint_C \log[X(z)]\, z^{n-1}\, dz$$

The contour of integration C is within the region of convergence.

The region of convergence must include the unit cycle for stability.

That‘s why we can use the inverse Fourier transform.

The real cepstrum is the inverse transform of the real part of $\hat{X}(e^{j\omega})$ and therefore is equal to the conjugate-symmetric part of $\hat{x}[n]$:

$$c_x[n] = \frac{\hat{x}[n] + \hat{x}^*[-n]}{2}$$

Alternative Expressions

We can use the logarithmic derivative to avoid the complex logarithm.

Assuming log[X(z)] is analytic, then

$$\hat{X}'(z) = \frac{X'(z)}{X(z)}$$

Using the z-transform formula we get

$$-n\,\hat{x}[n] = \frac{1}{2\pi j} \oint_C \frac{z\,X'(z)}{X(z)} z^{n-1}\,dz$$

Dividing both sides by (-n) yields

$$\hat{x}[n] = \frac{-1}{2\pi\, j\, n} \oint_C \frac{z\, X'(z)}{X(z)}\, z^{n-1}\, dz\,, \quad n \neq 0$$

and per definition

$$\hat{x}[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{X}(e^{j\omega})\, d\omega$$

A (nonlinear) difference equation for x[n] is :

$$x[n] = \sum_{k=-\infty}^{\infty} \frac{k}{n}\, \hat{x}[k]\, x[n-k], \quad n \neq 0$$

(for computation with recursive algorithms)

Complex cepstrum of exponential sequences

If a sequence x[n] consists of a sum of complex exponential sequences, it‘s z-transform is a rational function of z.

$$X(z)=\frac{A z^{r}\prod_{k=1}^{M_i}\left(1-a_k z^{-1}\right)\prod_{k=1}^{M_o}\left(1-b_k z\right)}{\prod_{k=1}^{N_i}\left(1-c_k z^{-1}\right)\prod_{k=1}^{N_o}\left(1-d_k z\right)}$$

i…inside, o…outside of the unit circle

then log[X(z)] is :

$$\hat{X}(z) = \log(A) + \log(z^r) + \sum_{k=1}^{M_i} \log\left(1 - a_k z^{-1}\right) + \sum_{k=1}^{M_o} \log\left(1 - b_k z\right)$$
$$- \sum_{k=1}^{N_i} \log\left(1 - c_k z^{-1}\right) - \sum_{k=1}^{N_o} \log\left(1 - d_k z\right)$$

A is real for real sequences but could be negative.

A nonzero r will cause a discontinuity in arg[X].

In practice this is avoided by determining A and r and altering the input so, that $A \rightarrow |A|$, $r \longrightarrow 0$

with the power series expansions

$$\log(1-\alpha z^{-1}) = -\sum_{n=1}^{\infty} \frac{\alpha^n}{n} z^{-n}, \quad |z| < |a|$$

$$\log(1-\beta z) = -\sum_{n=1}^{\infty} \frac{\beta^n}{n} z^{n}, \quad |z| < |\beta^{-1}|$$

we obtain

$$\hat{x}[n] = \begin{cases} \log|A|, & n = 0, \\ -\sum_{k=1}^{M_i} \frac{a_k^n}{n} + \sum_{k=1}^{N_i} \frac{c_k^n}{n}, & n > 0, \\ \sum_{k=1}^{M_o} \frac{b_k^{-n}}{n} + \sum_{k=1}^{N_o} \frac{d_k^{-n}}{n}, & n < 0. \end{cases}$$

From this we can derive the following general properties:

1) the complex cepstrum decays at least as fast as $1/|n|$

2) it has infinite duration, even if $x[n]$ has finite duration

3) it is real if $x[n]$ is real (poles and zeros are in complex conjugate pairs)

minimum-phase and maximum-phase sequences

A minimum-phase sequence is a real, causal and stable sequence whose poles and zeros are all inside the unit cycle.

That means that all the singularities of log[X(z)] are inside the unit cycle.

So we have the property

4) The complex cepstrum is causal (0 for all n<0) if and only if x[n] is minimum phase.

similarily we conclude, that the complex cepstrum for maximum-phase (=left sided) systems is also left-sided

5) The complex cepstrum is 0 for all $n>0$ if and only if $x[n]$ is maximum phase, i.e. $X(z)$ has all poles and zeros outside the unit circle

Hilbert Transform Relations

Causality (or anticausality) of a sequence places powerful constraints on the Fourier Transform.

The Fourier Transform of any real, causal sequence is almost completely determined by either the real or the imaginary part of the Fourier Transform.

Thus, if $\hat{x}[n]$ is causal (x[n] is minimum phase), we can obtain Hilbert transform relations between the real ($\log|X(e^{j\omega})|$) and imaginary ($\arg[X(e^{j\omega})]$) parts of the complex cepstrum.

As noted earlier, the cepstrum is the even part of the complex cepstrum:

$$c_x[n] = \frac{\hat{x}[n] + \hat{x}^*[-n]}{2}$$

So if the complex cepstrum is known to be causal, it can be recovered from the real cepstrum by frequency-invariant linear filtering.

$$\hat{x}[n] = c_x[n]\ \ell_{\min}[n]$$

$$\ell_{\min}[n] = 2u[n] - \delta[n]$$

Homomorphic deconvolution

- Generalized superposition

$T\{x_1[n]+x_2[n]\}= T\{x_1[n]\}+ T\{x_2[n]\}$

$T\{c\ x_1[n]\}=c\ T\{x_1[n]\}$

becomes

$T\{x_1[n] \square x_2[n]\}= T\{x_1[n]\} \circ T\{x_2[n]\}$

$T\{c ⌂ x_1[n]\}=c \blacksquare T\{x_1[n]\}$

with different input and output operations

Systems satisfying this generalized principle of superposition are called homomorphic systems since they can be represented by algebraically linear (homomorphic) mappings between input and output signal spaces.

Linear systems are obviously a special case where □ and ○ are addition and ⌂ and ▪ are multiplication.

Now suppose the input of a system is the convolution of two signals :

$x[n]=x_1[n]*x_2[n]$

Their Fourier transforms would therefore be multiplied :

$X(z)=X_1(z)\ X_2(z)$

applying the logarithm :

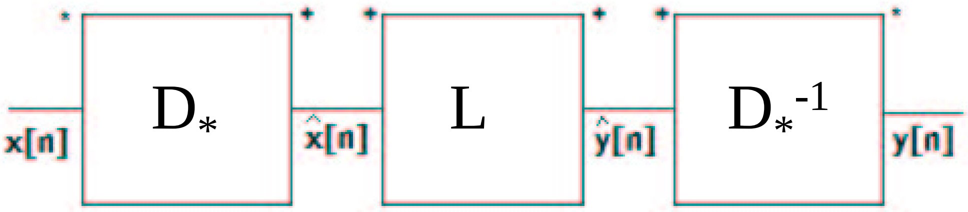
$\log[X(z)]=\log[X_1(z)] + \log[X_2(z)]$

So the cepstrum is:

$$\hat{x}[n] = \hat{x}_1[n] + \hat{x}_2[n]$$

Thus, the cepstrum can be interpreted as a System that satisfies the generalized principle of superposition with convolution as the input function of superposition and addition as the output function of superposition.

This system is called the Characteristic System for Convolution D_*.



L is a linear System in the usual sense.

Now if we choose a system L that removes the additive component $\hat{x}_2[n]$, then $x_2[n]$ will be removed from the convolutional combination.

In practical applications, L is a linear frequency-invariant system.

Such L are called ‚lifters' because of their similarity to filters in the frequency domain.

Applications

- processing signals containing echoes

  seismology, measuring properties of reflecting surfaces, loudspeaker design, dereverberation, restoration of acoustic recordings
- speech processing

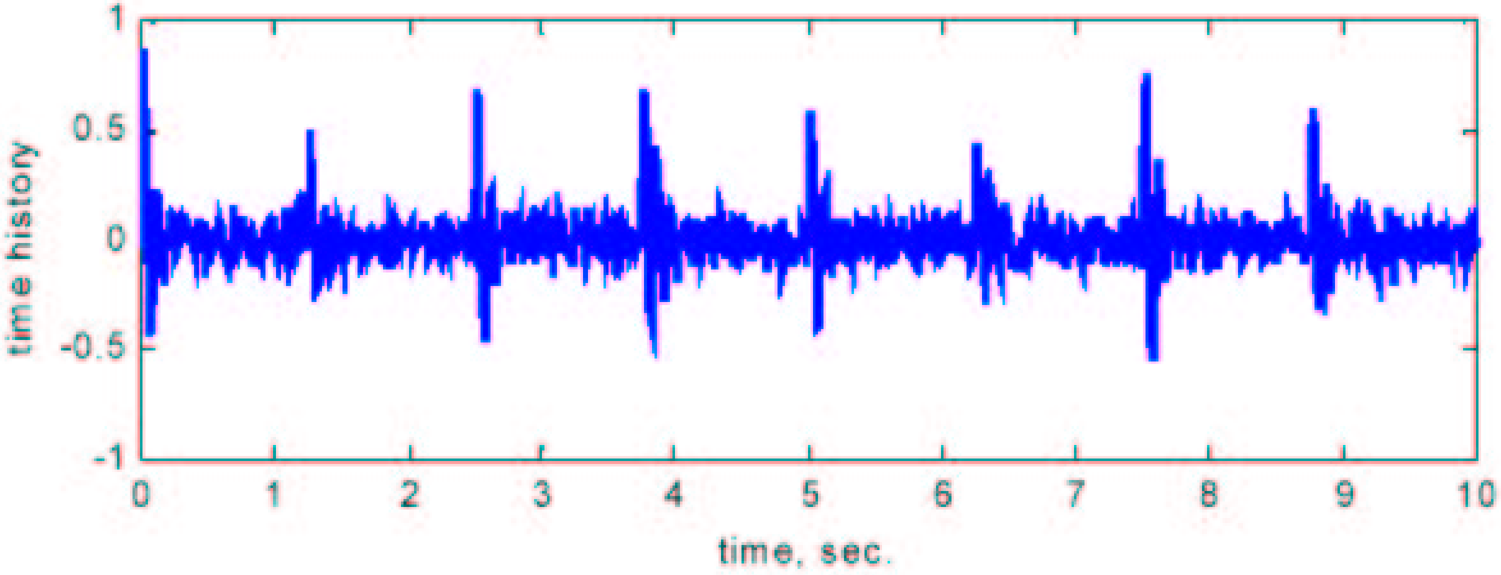
  estimating parameters of the speech model

- machine diagnostics

  detection of families of harmonics and sidebands, for example in gearbox and turbine vibrations

- calculating the minimum phase spectrum corresponding to a given log amplitude spectrum with Hilbert transform

example: delay time detection



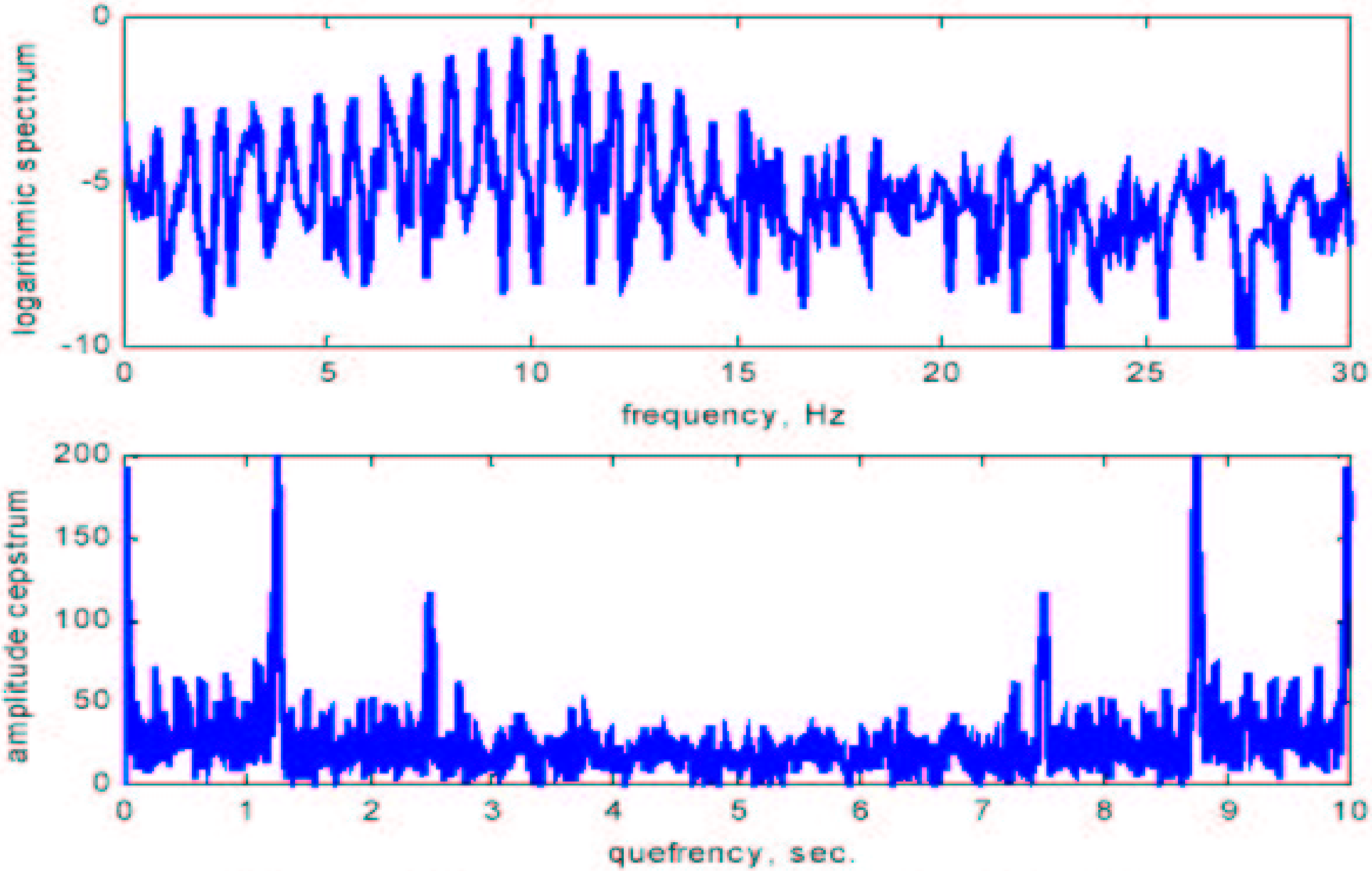
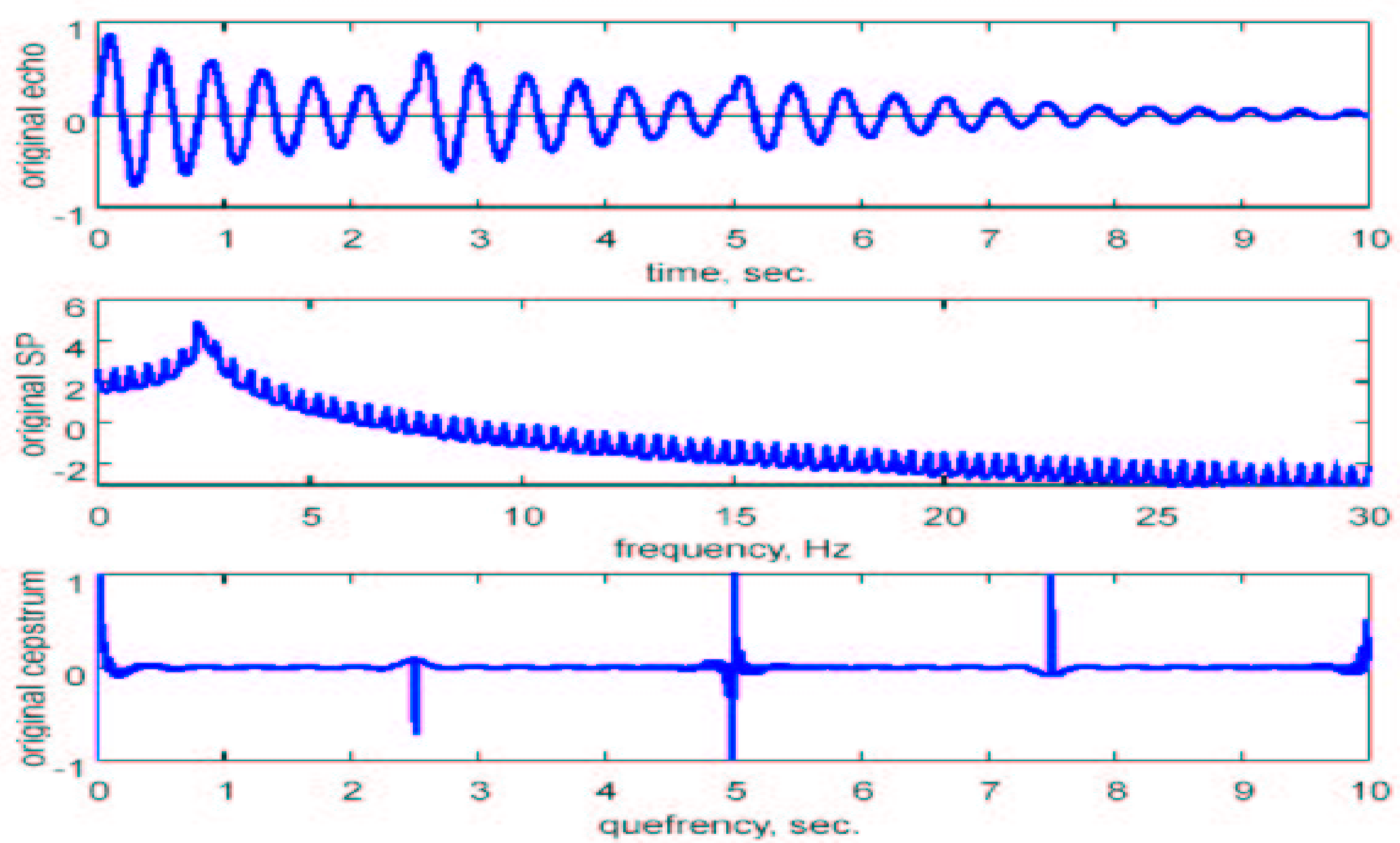


Figure A. Vibraton response of bearing with a fault:
(a) time history, (b) spectral density and (c) amplitude

Example: echo removal



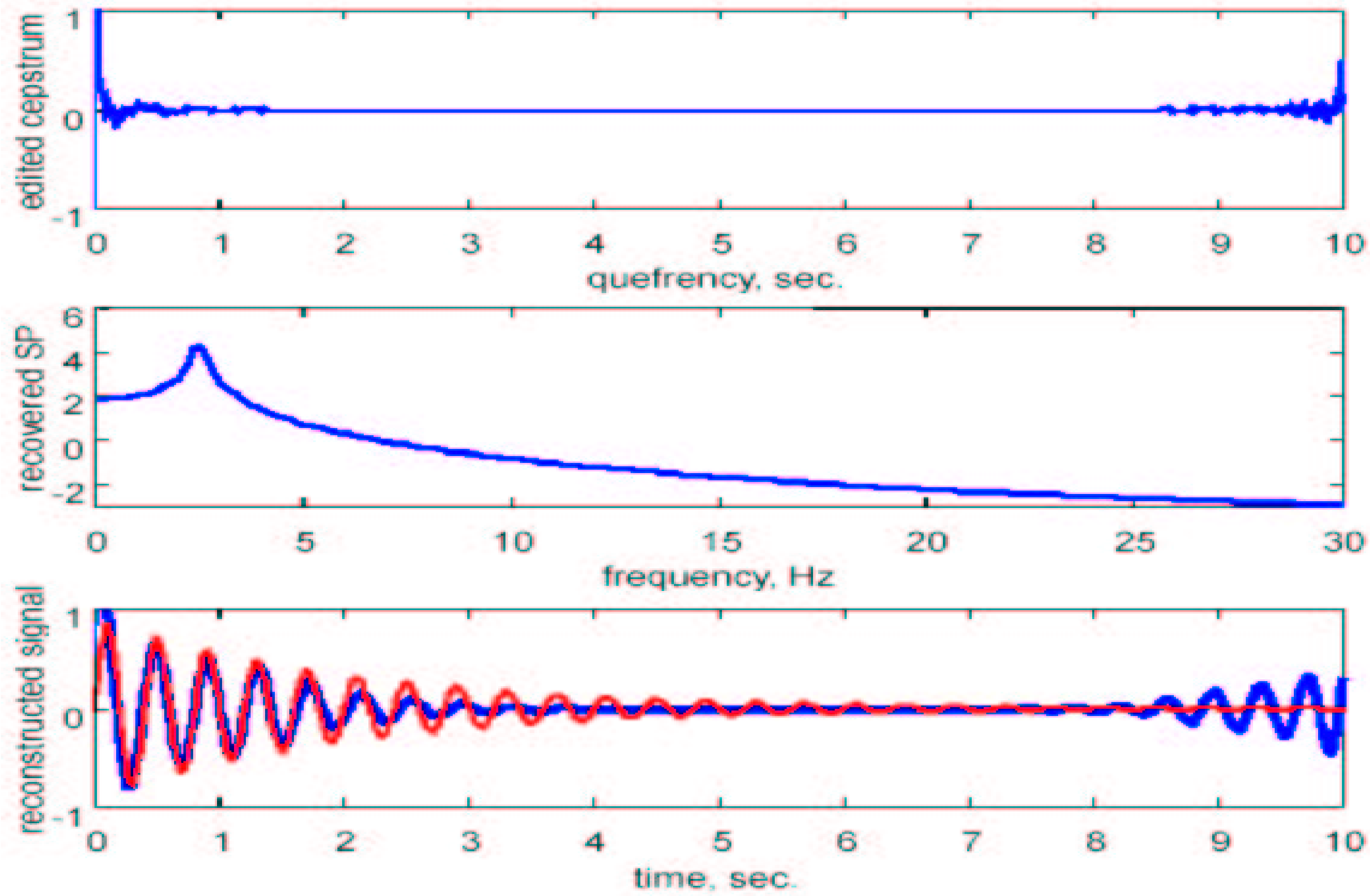


Figure B. Removal of echoes by using complex cepstrum.